THEORY OF RADIATION FROM THE OPEN END OF A CIRCULAR WAVEGUIDE FLUSH-MOUNTED TO A FLAT GROUND PLANE*

by

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1. *Introduction*

The problem of radiation into free space from the open end of a coaxial waveguide flush-mounted to an infinite ground plane has been considered previously by several investigators [Levine and Papas, 1951; and Cohn and Flesher, 1958]. In that problem, the dominant TEM mode is assumed incident in the waveguide. Because of the angular symmetry of the incident TEM mode and the geometry of the problem, the reflected and the radiated fields also remain angularly symmetric (i.e., $\partial/\partial\phi = 0$). Therefore, TE modes will not be excited and consequently, two scalar potentials, one inside and the other outside the waveguide, are needed to describe the electromagnetic fields everywhere, However, in the present problem (Figure 1), the waveguide is a circular one from which the dominant TE_{11} mode

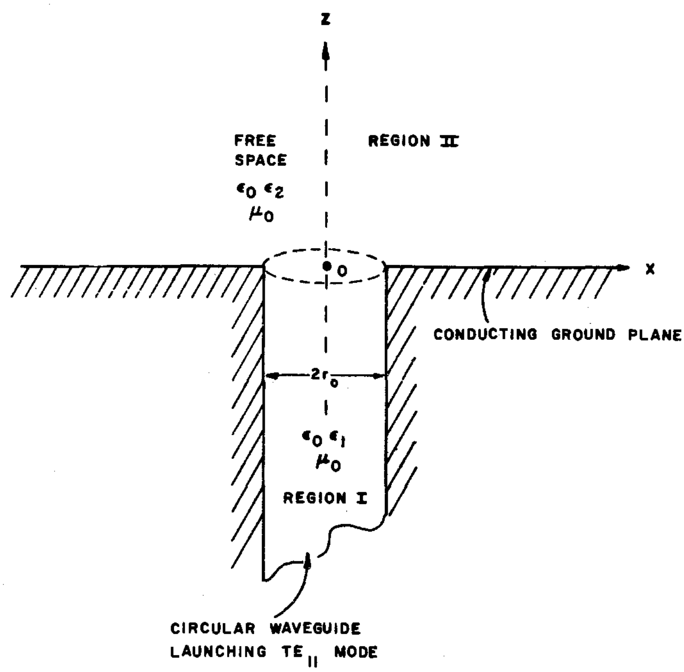

Figure 1. Radiation of TE_{11} Mode from the Open End of a Circular Waveguide Flush-Mounted to a Conducting Plane.

is launched. Since in this situation, the electromagnetic fields depend on the angular coordinate ϕ, the discontinuity of the open end of the waveguide will excite both TE and TM modes; in general, both inside and outside the waveguide. Therefore, in each region (waveguide and free-space), two potentials are necessary to represent the electromagnetic field. However, in a limiting situation which will be discussed in the text, only one scalar potential is needed for the description of the fields inside the waveguide in

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which all TM modes are assumed vanishingly small. Even in this situation where TM modes inside the waveguide are negligible, both TE and TM modes will be excited in the free-space (i.e., the unbounded half-space).

It is assumed that the circular waveguide is designed in such a way that it can carry only the dominant TE_{11} mode unattenuated. Even the TM_{11} mode becomes evanescent. Furthermore, it will also be supposed that the angular variation of all the excited electromagnetic modes is a linear combination of $\sin\phi$ and $\cos\phi$, such that they are consistent with the angular variation of the incident TE_{11} mode. This is a reasonable assumption as long as the geometry of the problem possesses angular symmetry. In particular, the angular dependence of the longitudinal magnetic field of the TE_{11} mode is chosen to be $\Phi_2(\phi) = \sigma_1 \cos\phi + i\,\sigma_2 \sin\phi$. The reason for this choice of the angular variation is that it is general for a TE_{11} mode and it can represent either linearly or elliptically polarized waves. It may be noted that circularly polarized waves do not exist in a circular waveguide. The real constants σ_1 and σ_2 depend on the ellipticity of the polarization. Then it turns out that the angular dependence of the longitudinal electric field of the excited $TM_{1\ell}$ mode must be given by $\Phi_1(\phi) = \sigma_1 \sin\phi - i\,\sigma_2 \cos\phi$. Therefore, the higher ordered modes excited inside the waveguide correspond to different radial variation of the fields.

In order to express the electromagnetic fields in the unbounded half-space, an appropriate Hankel transform has been introduced. The radiation fields (i.e., far-fields), which thus have integral representations, are evaluated formally by the method of saddle point integration. The amplitudes of the excited fields and the reflection coefficient which is related to the admittance of the circular aperture depend on the unknown radial and angular electric fields on the aperture.

These unknown electric fields on the aperture can be shown to satisfy two simultaneous integral equations which have been solved by two methods: (1) variational principle and (2) successive iteration method. For the first order approximation, it has been possible to show that these two methods lead to identical results.

Assuming from the beginning that the higher order TE modes as well as the TM modes are not excited by the discontinuity of the circular aperture, Mishustim [1965, a Russian work] and Bailey, Samaddar and Swift [1967] in a joint work independently calculated the input admittance of the aperture and the radiated fields (for which $\sigma_1 = 1$ and $\sigma_2 = 0$). These results are also obtained as a special case (lowest order approximation) in this paper. It may be noted that the English translation of the work of Mishustim is not available yet. Also, the method of approach adopted by Mishustim seems to be different from the present work as well as that Bailey, et al.

It is shown that the general expression for the admittance does not depend either on the nature of polarization (linear or elliptical) or on the amplitude of the incident wave.

2. *Formulation of the problem*

The geometry of the problem is shown in Figure 1. A circular waveguide, fed in the dominant TE_{11} mode, which is assumed to be elliptically polarized, is flush-mounted to an infinite, perfectly conducting flat-grounded plane. The relative dielectric constants of the materials inside and outside (unbounded half-space) the waveguide are taken to be ϵ_1 and ϵ_2, whereas the relative permeability is assumed unity everywhere. Because of the cylindrical symmetry of the problem, cylindrical coordinates r, ϕ, z will be chosen, with the origin at the center of the circular aperture. The waveguide region and the free half-space correspond to $z < 0$ and $z > 0$ respectively.

It is easy to show that inside the waveguide, both the longitudinal (or axial) fields, E_{z1} and H_{z1}, satisfy the following Helmholtz equation (the assumed harmonic time dependent $e^{-i\omega t}$ will be suppressed throughout the analysis):

$$(\nabla^2 + K_1^2)\, E_{z1} = 0 \tag{1i}$$

$$(\nabla^2 + K_1^2)\, H_{z1} = 0 \tag{1ii}$$

where,

$$K_1^2 = K_0^2 \epsilon_1 = \omega^2 \mu_o \epsilon_o \epsilon_1 \tag{1iii}$$

The transverse components (i.e., r and ϕ components) of the electromagnetic fields can be expressed in terms of E_{z1} and H_{z1} in the following manner:

$$(K_1^2 + \frac{\partial^2}{\partial z^2})\, \underline{E}_{t1} = \frac{\partial}{\partial z} \nabla_t E_{z1} + i\omega\mu_o \nabla_t H_{z1} \times \hat{z}_0 \tag{2i}$$

$$(K_1^2 + \frac{\partial^2}{\partial z^2})\, \underline{H}_{t1} = \frac{\partial}{\partial z} \nabla_t H_{z1} + i\omega\mu_o \epsilon_1\, \hat{z}_o \times \nabla_t E_{z1} \tag{2ii}$$

If the vector $\underline{P}$ stands for either $\underline{E}$ or $\underline{H}$, then one may write $\underline{P} = \underline{P}_t + \hat{z}_o P_z$ and similarly one has $\nabla = \nabla_t + \hat{z}_o\, \partial/\partial z$, where $\hat{z}_o$ is the unit vector in the z-direction.

In order to represent the electromagnetic fields in the free-space (outside the waveguide), one may express all the relevent quantities in the way shown in the relations (1) to (2), by replacing the subscript 1 by 2. However, for convenience, which should be clear in the text, two Hertz potentials F_1 and F_2 (Stratton, 1941) are introduced in the following fashion:

$$(\nabla^2 + K_2^2)\, F_1 = 0 \tag{3i}$$

$$(\nabla^2 + K_2^2)\, F_2 = 0 \tag{3ii}$$

$$E_{z2} = (\frac{\partial^2}{\partial z^2} + K_2^2)\, F_1 \tag{4i}$$

$$E_{r2} = i\omega\mu_o \frac{1}{r} \frac{\partial}{\partial \phi} F_2 + \frac{\partial^2 F_1}{\partial r\, \partial z} \tag{4ii}$$

$$E_{\phi 2} = -i\omega\mu_o \frac{\partial}{\partial r} F_2 + \frac{1}{r} \frac{\partial^2 F_1}{\partial \phi\, \partial z} \tag{4iii}$$

$$H_{z2} = (\frac{\partial^2}{\partial z^2} + K_2^2)\, F_2 \tag{4iv}$$

$$H_{r2} = \frac{\partial^2 F_2}{\partial r\, \partial z} - i\omega\epsilon_o \epsilon_2 \frac{1}{r} \frac{\partial}{\partial \phi} F_1 \tag{4v}$$

$$H_{\phi 2} = \frac{1}{r} \frac{\partial^2 F_2}{\partial \phi\, \partial z} + i\omega\epsilon_o \epsilon_2 \frac{\partial}{\partial r} F_1 \tag{4vi}$$

where,

$$K_2^2 = K_o^2 \epsilon_2$$

Since the incident TE_{11} mode inside the waveguide is considered elliptically polarized, the ϕ-dependence of H_z can be taken to be $\Phi_2(\phi) = \sigma_1 \cos\phi + i\sigma_2 \sin\phi$. Consequently, the angular dependence of the excited longitudinal field E_z becomes $\Phi_1(\phi) = \sigma_1 \sin\phi - i\sigma_2 \cos\phi$. The respective ϕ-dependence of H_z and E_z will be the same both inside and outside the waveguide. These requirements follow from the single-valued behavior of the electromagnetic fields.

If Γ be the reflection coefficient, then the total fields (incident and reflected) inside the waveguide can be expressed in the following manner:

$$H_{z1} = \Phi_2(\phi)\left[A_o J_1(\eta_1 r)\, e^{i\beta_1 z}\left\{1 + \Gamma e^{-2i\beta_1 z}\right\} - \sum_{n=2}^{\infty} A_n J_1(\eta_n r)\, e^{\alpha_n z}\right], \tag{5i}$$

$$E_{z1} = \Phi_1(\phi) \sum_{\ell=1}^{\infty} B_\ell J_1(\lambda_\ell r)\, e^{\gamma_\ell z}, \tag{5ii}$$

$$E_{r1} = -i\,\Phi_1(\phi) \frac{\omega\mu_o A_o}{\eta_1^2 r} J_1(\eta_1 r)\, e^{i\beta_1 z}\left\{1 + \Gamma e^{-2i\beta_1 z}\right\} - \tag{6i}$$
$$-\omega\mu_o \sum_{n=2}^{\infty} \frac{A_n}{\eta_n^2 r} J_1(\eta_n r)\, e^{\alpha_n z} + i \sum_{\ell=1}^{\infty} \frac{B_\ell \gamma_\ell}{\lambda_\ell} J_1'(\lambda_\ell r)\, e^{\gamma_\ell z},$$

$$E_{\phi 1} = i\Phi_2(\phi)\left[-\frac{\omega\mu_o A_o}{\eta_1} J_1'(\eta_1 r) e^{i\beta_1 z}\left\{1 + \Gamma e^{-2i\beta_1 z}\right\} + \right. \tag{6ii}$$
$$\left. + \omega\mu_o \sum_{n=2}^{\infty} \frac{A_n}{\eta_n} J_1'(\eta_n r)\, e^{\alpha_n z} - i \sum_{\ell=1}^{\infty} \frac{B_\ell \gamma_\ell}{\lambda_\ell^2 z} J_1(\lambda_\ell r)\, e^{\gamma_\ell z}\right],$$

$$H_{r1} = i\Phi_2(\phi)\left[\frac{\beta_1 A_o}{\eta_1} J_1'(\eta_1 r)\, e^{i\beta_1 z}\left\{1 - \Gamma e^{-2i\beta_1 z}\right\} + \right. \tag{6iii}$$
$$\left. + i \sum_{n=2}^{\infty} \frac{A_n \alpha_n}{\eta_n} J_1'(\eta_n r)\, e^{\alpha_n z} - \omega\epsilon_o \epsilon_1 \sum_{\ell=1}^{\infty} \frac{B_\ell J_1(\lambda_\ell r)}{\lambda_\ell^2 r} e^{\gamma_\ell z}\right],$$

$$H_{\phi 1} = -i\Phi_1(\phi)\left[\frac{\beta_1 A_o}{\eta_1^2 r} J_1(\eta_1 r)\, e^{i\beta_1 z}\left\{1 - \Gamma e^{-2i\beta_1 z}\right\} + \right. \tag{6iv}$$
$$\left. + i \sum_{n=2}^{\infty} \frac{A_n \alpha_n}{\eta_n^2 r} J_1(\eta_1 r)\, e^{\alpha_n z} - \omega\epsilon_o \epsilon_1 \sum_{\ell=1}^{\infty} \frac{B_\ell}{\lambda_\ell} J_1'(\lambda_\ell r)\, e^{\gamma_\ell z}\right],$$

where,

$$\left.\begin{aligned}
\beta_1 &= \sqrt{K_1^2 - \eta_1^2} \\
\alpha_n &= \sqrt{\eta_n^2 - K_1^2}, \quad n \geq 2 \\
J_1'(\eta_n r_o) &= 0, \quad \text{for } n = 1, 2, 3, \ldots \\
r_o &= \text{radius of the waveguide}
\end{aligned}\right\} \quad \text{for TE modes} \tag{7}$$

$$\left.\begin{aligned} \gamma_\ell &= \sqrt{\lambda_\ell^2 - K_1^2} \\ J_1(\lambda_\ell r_o) &= 0, \quad \ell = 1, 2, \ldots \end{aligned}\right\} \quad \text{for TM modes} \tag{8}$$

The quantitites σ_1 and σ_2 appearing in $\Phi_1(\phi)$ and $\Phi_2(\phi)$ together with A_0 are assumed to be known and they related to the amplitude and the ellipticity of the incident TE_{11} mode. The amplitude coefficients A_n and B_ℓ for the higher ordered modes as well as the reflection coefficient Γ are to be determined from the boundary conditions at $z = 0$ (at the aperture). Note that the requirements, $J_1'(\eta_n r_o) = 0$ and $J_1(\lambda_\ell r_o) = 0$, which determine η_n and λ_ℓ respectively, for a given r_o, ensure the vanishing of the tangential electric fields (and hence the normal component of the magnetic fields) at the wall ($r = r_0$) of the waveguide.

Because of the cylindrical symmetric of the problem, it is expedient to introduce Hankel transform of the differential equations (3) to (4) appropriate for the free-space fields, with respect to the Kernel $J_1(\zeta r)$, where ζ is the transform variable. Therefore, if Q_i represents a well behaved function of r and z, then its Hankel transform pair is defined in the following manner [Morse and Feshbach, 1953]:

$$Q_i(r, z) = \int_0^\infty \zeta \; J_1(\zeta r) \; \hat{Q}_i(\zeta, z) \, d\zeta \tag{9i}$$

where,

$$\hat{Q}_i(\zeta, z) = \int_0^\infty r \; J_1(\zeta r) \; Q_i(r, z) \, dr \tag{9ii}$$

Then, if $\hat{Q}_i(\zeta, z)$ is an odd function of ζ, then (9i) may be rewritten as [Samaddar, 1965]:

$$Q_i(r, z) = \frac{1}{2} \int_{\infty e^{i\pi}}^{\infty} \zeta \; H_1^{(1)}(\zeta r) \; \hat{Q}_i(\zeta, z) \, d\zeta, \quad \mathrm{Im}\zeta > 0; \tag{10}$$

Now, in particular, if we assume that $\zeta \hat{Q}_i(\zeta, z) = T_i(\zeta) \, e^{i\xi z}$ where $\xi = (K_2^2 - \zeta^2)^{1/2}$, then following Equation (16) of Samaddar (1965), the asymptotic value of (saddle-point contribution) (10) for $|\zeta r| >> 1$ can be expressed in the following manner [for $\sin\theta \neq 0$]:

$$Q_i(r, z) \sim - \left[\cot\theta \; T_i(\zeta)\right] \frac{e^{iK_2 R}}{R} \tag{11}$$

in which the following substitutions are inderstood:

$$\left.\begin{aligned} \zeta &= K_2 \sin\theta \\ \xi &= K_2 \cos\theta \\ z &= R \cos\theta \\ r &= R \sin\theta \end{aligned}\right\} \tag{12}$$

Therefore, if one represents $F(r, \phi, z)$ by,

$$F_i(r, \phi, z) = Q_i(r, z) \; \Phi_i(\phi), \quad \text{for } i = 1, 2, \tag{13}$$

the asymptotic values of the Hertz potentials F_1 and F_2 can readily be obtained from Equations (11) and (13).

Now, employing the above-mentioned Hankel transform, the following integral representations of the electromagnetic fields in the free-space region can be established.

$$F_1 = \Phi_1(\phi) \int_0^\infty \zeta\ J_1(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \tag{14i}$$

$$F_2 = \Phi_2(\phi) \int_0^\infty \zeta\ J_1(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta \tag{14ii}$$

$$H_{z2} = \Phi_2(\phi) \int_0^\infty \zeta^3\ J_1(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta \tag{15i}$$

$$E_{z2} = \Phi_1(\phi) \int_0^\infty \zeta^3\ J_1(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \tag{15ii}$$

$$E_{r2} = -i\ \Phi_1(\phi) \left[\frac{\omega\mu_o}{r} \int_0^\infty \zeta\ J_1(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta - \int_0^\infty \xi\ \zeta^2\ J_1'(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \right], \tag{16i}$$

$$E_{\phi 2} = i\ \Phi_2(\phi) \left[-\ \omega\mu_o \int_0^\infty \zeta^2 J_1'(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta + \frac{1}{r} \int_0^\infty \xi\ \zeta\ J_1(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \right], \tag{16ii}$$

$$H_{r2} = i\ \Phi_2(\phi) \left[\int_0^\infty \xi\ \zeta^2\ J_1'(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta - \frac{\omega\ \epsilon_o \epsilon_2}{r} \int_0^\infty \zeta\ J_1(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \right], \tag{16iii}$$

$$H_{\phi 2} = -i\ \Phi_1(\phi) \left[\frac{1}{r} \int_0^\infty \xi\ \zeta\ J_1(\zeta r)\ T_2(\zeta)\ e^{i\xi z}\ d\zeta - \omega\ \epsilon_o \epsilon_2 \int_0^\infty \zeta^2\ J_1'(\zeta r)\ T_1(\zeta)\ e^{i\xi z}\ d\zeta \right]. \tag{16iv}$$

The amplitude factors $T_1(\zeta)$ and $T_2(\zeta)$ will be determined from the boundary conditions at $z = 0$.

3. Boundary conditions

The unknown coefficients, Γ, A_n, B_ℓ, $T_1(\zeta)$ and $T_2(\zeta)$ appearing in the relations (5), (6) and (14) to (16) are to be determined from the requirement of continuity of the tangential electromagnetic fields at the aperture (at $z = 0$). These continuity relations are expressed in the following manner:

$$E_{r1}\Big|_{z=0} = E_{r2}\Big|_{z=0} = \mathcal{E}_1(r)\ \Phi_1(\phi) \tag{17i}$$

$$E_{\phi 1}\Big|_{z=0} = E_{\phi 2}\Big|_{z=0} = \mathcal{E}_2(r)\ \Phi_2(\phi) \tag{17ii}$$

$$H_{r1}\Big|_{z=0} = H_{r2}\Big|_{z=0} = \mathcal{H}_1(r)\ \Phi_2(\phi) \tag{17iii}$$

$$H_{\phi 1}\Big|_{z=0} = H_{\phi 2}\Big|_{z=0} = \mathcal{H}_2(r)\ \Phi_1(\phi) \tag{17iv}$$

The radial distributions, $\mathcal{E}_1(r)$, $\mathcal{E}_2(r)$, $\mathcal{H}_1(r)$ and $\mathcal{H}_2(r)$ of the respective aperture fields are understood to be defined by (17). These unknown aperture distributions are related to the unknown coefficients, Γ, A_n, B_ℓ, $T_1(\zeta)$ and $T_2(\zeta)$. The constants Γ and A_n are to be determined from similar types of operations (orthogonality properties of $J_1(\eta_n r)$). Though A_n is defined for $n \geq 2$, one could have defined it also for $n = 1$, in which case A_1 would have been related to Γ. Therefore, there should not be any confusion from the apparent appearance of five unknown quantities which are to be determined from four boundary conditions specified by (17).

Imposing the boundary conditions (17) on (6) and (16), the following relations are obtained:

$$\frac{\omega\mu_o A_o}{\eta_1^2 r}(1+\Gamma)\ J_1(\eta_1 r) - \omega\mu_o \sum_{n=2}^{\infty} \frac{A_n}{\eta_n^2 r} J_1(\eta_n r) + i\sum_{\ell=1}^{\infty} \frac{B_\ell \gamma}{\lambda_\ell} J_1'(\lambda_\ell r) =$$

$$= i\mathcal{E}_1(r) = \tag{18}$$

$$= \frac{\omega\mu_o}{r}\int_0^\infty d\zeta\ \zeta\ J_1(\zeta r)\ T_2(\zeta) - \int_0^\infty d\zeta\ \zeta^2\ \xi\ J_1'(\zeta r)\ T_1(\zeta),$$

$$\frac{\omega\mu_o A_o}{\eta_1}(1+\Gamma)\ J_1'(\eta_1 r) - \omega\mu_o \sum_{n=2}^{\infty} \frac{A_n}{\eta_n} J_1'(\eta_n r) + i\sum_{\ell=1}^{\infty} \frac{B_\ell \gamma_\ell}{\lambda_\ell^2 r} J_1(\lambda_\ell r) =$$

$$= i\ \mathcal{E}_2(r) = \tag{19}$$

$$= \omega\mu_o \int_0^\infty d\zeta\ \zeta^2\ J_1'(\zeta r)\ T_2(\zeta) - \frac{1}{r}\int_0^\infty d\zeta\ \zeta\ \xi\ J_1(\zeta r)\ T_1(\zeta),$$

$$\frac{\beta_1 A_o}{\eta_1}(1-\Gamma)\ J_1'(\eta_1 r) + i\sum_{n=2}^{\infty} \frac{A_n \alpha_n}{\eta_n} J_1'(\eta_n r) - \omega\ \epsilon_o \epsilon_1 \sum_{\ell=1}^{\infty} \frac{B_\ell J_1(\lambda_\ell r)}{\lambda_\ell^2 r} =$$

$$= -i\ \mathcal{H}_1(r) = \tag{20}$$

$$= \int_0^\infty d\zeta\ \xi\ \zeta^2\ J_1'(\zeta r)\ T_2(\zeta) - \frac{\omega\epsilon_o \epsilon_2}{r}\int_0^\infty d\zeta\ \zeta\ J_1(\zeta r)\ T_1(\zeta),$$

$$\frac{\beta_1 A_o}{\eta_1^2 r}(1-\Gamma)\ J_1(\eta_1 r) + i\sum_{n=2}^{\infty} \frac{A_n \alpha_n}{\eta_1^2 r} J_1(\eta_n r) - \omega\ \epsilon_0 \epsilon_1 \sum_{\ell=1}^{\infty} \frac{B_\ell}{\lambda_\ell} J_1'(\lambda_\ell r) =$$

$$= i\, \mathcal{H}_2(r) = \tag{21}$$

$$= \frac{1}{r}\int_0^\infty d\zeta\; \xi\; \zeta\; J_1(\zeta r)\; T_2(\zeta) - \omega\epsilon_0\epsilon_2 \int_0^\infty d\zeta\; \zeta^2\; J_1'(\zeta r)\; T_1(\zeta).$$

Now, multiplying Equation (18) by r d/dr $J_1(\zeta r)$ dr and Equation (19) by $J_1(\zeta r)$ dr respectively and then integrating the respective left-hand side from 0 to r_0 and the right-hand side from 0 to ∞, one finds the following relation after adding these two results (using the relations (A-10) to (A-13)). Before performing the integration with respect to r, the integration variable, ζ, on the right-hand sides of (18) and (19) is changed to ζ' for convenience.

$$\int_0^{r_0} dr \left\{ \mathcal{E}_1(r)\; r\, \frac{d}{dr}\, J_1(\zeta r) + \mathcal{E}_2(r)\; J_1(\zeta r) \right\} = i\, \zeta^2\, \xi\; T_2(\zeta) =$$

$$= \left[-i\omega\mu_0 \frac{A_0(1+\Gamma) J_1(\eta_1 r_0)}{\eta_1^{\,2}} + i\omega\mu_0 \sum_{n=2}^{\infty} \frac{A_n}{\eta_n^{\,2}} J_1(\eta_n r_0) + \right. \tag{22}$$

$$\left. + \sum_{\ell=1}^{\infty} \frac{B_\ell \gamma_\ell \quad r_0 \zeta^2 J_0(\lambda_\ell r_0)}{\lambda_\ell (\zeta^2 - \lambda_\ell^{\,2})} \right] J_1(\zeta r_0)$$

Similarly, first changing the integration variable on the right-hand side of (18) and (19) from ζ to ζ', we multiply (18) and (19) by $J_1(\zeta r)$ dr and r d/dr $J_1(\zeta r)$ dr respectively. Then, integrating the left-hand side from 0 to r_0 and the right-hand side from 0 to ∞, we have (using (A-12), (A-13) and (A-15)):

$$\int_0^{r_0} dr \left\{ \mathcal{E}_1(r)\; J_1(\zeta r) + \mathcal{E}_2(r)\, \frac{d}{dr}\, J_1(\zeta r) \right\} = -i\omega\mu_0\; \zeta^2\; T_2(\zeta) =$$

$$= -i\omega\mu_0 \left[\frac{A_0(1+\Gamma)\; J_1(\eta_1 r_0)}{(\eta_1^{\,2} - \zeta^2)} - \sum_{n=2}^{\infty} \frac{A_n J_1(\eta_n r_0)}{(\eta_n^{\,2} - \zeta^2)} \right] \zeta\; r_0 J_1'(\zeta r\;). \tag{23}$$

Further appropriate operations on the relations (18) to (21) will depend on the approach (namely variational method or successive iteration method) by which those integral equations are solved.

4. *Variational principle*

For the formulation of a variational principle, first of all expressions for A_0, A_n, B_ℓ, $T_1(\zeta)$ and $T_2(\zeta)$ must be obtained in terms of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ from (18) and (19). When these expressions are substituted in (20) and (21), there results two simultaneous integral equations involving $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$. From these two integral equations, a variational expression for the aperture admittance will be obtained.

The relations (22) and (23) can be used as expressions for $T_1(\zeta)$ and $T_2(\zeta)$ respectively in terms of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$. Now, in order to represent $A_0(1+\Gamma)$ and A_n in terms of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$, let us multiply (18) and (19) by $J_1(\eta_n r)$ dr and r d/dr $J_1(\eta_n r)$ dr respectively. Then, integrating from 0 to r_0, we have (using (A-12) and (A-14)):

$$A_0(1+\Gamma) = \frac{i2\,\eta_1^{\,2}}{\omega\mu_o\left[(\eta_1 r_o)^2 - 1\right] J_1^{\,2}(\eta_1 r_o)} \int_0^{r_o} dr \left\{ \mathcal{E}_1(r)\, J_1(\eta_1 r) + \right.$$

$$\left. + \mathcal{E}_2(r)\, r \frac{d}{dr} J_1(\eta_1 r) \right\} = \frac{2\,\eta_1^{\,4} r_o}{\left[(\eta_1 r_o)^2 - 1\right] J_1(\eta_1 r_o)} \times \tag{24}$$

$$\times \left[\int_0^{\infty} \frac{d\zeta\; \zeta^2 J_1'(\zeta r_o)\; T_2(\zeta)}{\eta_1^{\,2} - \zeta^2} - \frac{1}{\omega\mu_o \eta_1^{\,2} r_o} \int_0^{\infty} \zeta\; \xi\; \hat{T}_1(\zeta) e^{i\xi_o h} J_1(\zeta r_o) d\zeta \right],$$

$$A_n = -\frac{i2\,\eta_n^{\,2}}{\omega\mu_o\left[(\eta_n r_o)^2 - 1\right] J_1^{\,2}(\eta_n r_o)} \int_0^{r_o} dr \left\{ \mathcal{E}_1(r)\, J_1(\eta_n r) + \right.$$

$$\left. + \mathcal{E}_2(r)\, r \frac{d}{dr} J_1(\eta_n r) \right\} = -\frac{2\,\eta_n^{\,4} r_o}{\left[(\eta_n r_o)^2 - 1\right] J_1^{\,2}(\eta_n r_o)} \times \tag{25}$$

$$\times \left[\int_0^{\infty} \frac{d\zeta\; \zeta^2 J_1'(\zeta r_o)\; T_2(\zeta)}{\eta_n^{\,2} - \zeta^2} - \frac{1}{\omega\mu_o \eta_n^{\,2} r_o} \int_0^{\infty} \zeta\; \xi\; \hat{T}_1(\zeta) e^{i\xi_o h}\; J_1(\zeta r_o) d\zeta \right],$$

for $n \geqq 2$.

Finally, to express B_ℓ in terms of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$, change the summation index ℓ to ℓ' and then multiply (18) and (19) by $r\, d/dr\; J_1(\lambda_\ell r)\, dr$ and $J_1(\lambda_\ell r)\, dr$ respectively. Then, integrating from 0 to r_0, we find (using (A-11), (A-12) and (A-16)):

$$B_\ell = \frac{2}{\gamma_\ell r_o^{\,2} J_o^{\,2}(\lambda_\ell r_o)} \int_0^{r_o} dr \left\{ \mathcal{E}_1(r)\, r \frac{d}{dr} J_1(\lambda_\ell r) + \mathcal{E}_2(r)\, J_1(\lambda_\ell r) \right\} =$$

$$= \frac{2i\,\lambda_\ell}{r_o \gamma_\ell J_o(\lambda_\ell r_o)} \int_0^{\infty} \frac{d\zeta\; \zeta^3 \xi\; T_1(\zeta)\; J_1(\zeta r_o)}{\zeta^2 - \lambda_\ell^{\,2}}\,. \tag{26}$$

It may be noted here that the last equality in all of the preceding relations (24) to (26) will not be needed. However, they are presented here for the sake of completeness only.

Now, substituting the values of $T_1(\zeta)$, $T_2(\zeta)$, A_o, A_n and B_ℓ in terms of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ from (32) to (26) into (20) and (21), the following simultaneous integral equations are obtained:

$$\Lambda\; L_0 \frac{d}{dr} J_1(\eta_1 r) = L_1(r,r')\; \mathcal{E}_1(r') + L_2(r,r')\; \mathcal{E}_2(r'), \tag{27}$$

$$\Lambda\; L_0 \frac{J_1(\eta_1 r)}{r} = L_3(r,r')\; \mathcal{E}_1(r') + L_4(r,r')\; \mathcal{E}_2(r'). \tag{28}$$

where we have,

$$\Lambda = \frac{2\,Y(0)}{\left[(\eta_1 r_o)^2 - 1\right] J_1^2(\eta_1 r_o)}, \tag{29}$$

$$Y(0) = \frac{\beta_1(1+\Gamma)}{\omega\mu_o(1+\Gamma)} = \text{aperture admittance}, \tag{30}$$

$$L = \int_0^{r_o} dr' \left\{ \mathcal{E}_1(r') J_1(\eta_1 r') + \mathcal{E}_2(r') r' \frac{d}{dr'} J_1(\eta_1 r') \right\}, \tag{31}$$

The operators are defined as;

$$L_1(r,r')\mathcal{E}_1(r') = \int_0^{r_o} dr' \left\{ \mathcal{E}_1(r') \right\} \left[\sum_{n=2}^{\infty} \frac{2i\alpha_n J_1(\eta_n r') \frac{d}{dr} J_1(\eta_n r)}{\omega\mu_o \left|(\eta_n r_o)^2 - 1\right| J_1^2(\eta_n r_o)} - \right.$$

$$-2i\omega\epsilon_o \epsilon_1 \sum_{\ell=1}^{\infty} \frac{r' \frac{d}{dr'} J_1(\lambda_\ell r') J_1(\lambda_\ell r)/r}{(\lambda_\ell r_o)^2 \gamma_\ell J_0^2(\lambda_\ell r_o)} + \tag{32}$$

$$\left. + \int_0^{\infty} \frac{d\zeta\,\xi}{\omega\mu_o\zeta} J_1(\zeta r') \frac{d}{dr} J_1(\zeta r) + \omega\epsilon_0\epsilon_2 \int_0^{\infty} \frac{d\zeta r' \frac{d}{dr'} J_1(\zeta r')}{\zeta\ \xi} \cdot \frac{J_1(\zeta r)}{r} \right],$$

$$L_2(r,r')\mathcal{E}_2(r') = \int_0^{r_o} dr' \left\{ \mathcal{E}_2(r') \right\} \left[\sum_{n=2}^{\infty} \frac{2i\alpha_n r' \frac{d}{dr'} J_1(\eta_n r')}{\omega\mu_o \left[(\eta_n r_o)^2 - 1\right] J_1^2(\eta_n r_o)} \frac{d}{dr} J_1(\eta_n r) - \right.$$

$$- 2i\omega\epsilon_o\epsilon_1 \sum_{\ell=1}^{\infty} \frac{J_1(\lambda_\ell r') J_1(\lambda_\ell r)/r}{(\lambda_\ell r_o)^2 \gamma_\ell \quad J_o^2(\lambda_\ell r_o)} + \tag{33}$$

$$\left. + \int_0^{\infty} \frac{d\zeta\,\xi}{\omega\mu_o\zeta} r' \frac{d}{dr'} J_1(\zeta r') \frac{d}{dr} J_1(\zeta r) + \omega\,\epsilon_o\epsilon_2 \int_0^{\infty} \frac{d\zeta}{\zeta\xi} J_1(\zeta r') J_1(\zeta r)/r \right],$$

$$L_3(r,r')\mathcal{E}_1(r') = \int_0^{r_o} dr' \left\{ \mathcal{E}_1(r') \right\} \left[\sum_{n=2}^{\infty} \frac{2i\alpha_n J_1(\eta_n r') J_1(\eta_n r)/r}{\omega\mu_o \left[(\eta_n r_o)^2 - 1\right] J_1^2(\eta_n r_o)} - \right.$$

$$- 2i\omega\epsilon_o\epsilon_1 \sum_{\ell=1}^{\infty} \frac{r' \frac{d}{dr'} J_1(\lambda_\ell r') \frac{d}{dr} J_1(\lambda_\ell r)}{(\lambda_\ell r_o)^2 \gamma_\ell \quad J_o^2(\lambda_\ell r_o)} + \tag{34}$$

$$\left. + \int_0^{\infty} \frac{d\zeta\,\xi}{\omega\mu_o\zeta} J_1(\zeta r') J_1(\zeta r)/r + \omega\epsilon_o\epsilon_2) \int_0^{\infty} \frac{d\zeta r' \frac{d}{dr'} J_1(\zeta r') \frac{d}{dr} J_1(\zeta r)}{\zeta\ \xi} \right],$$

$$L_4(r,r')\mathcal{E}_2(r') = \int_0^{r_o} dr' \left\{ \mathcal{E}_2(r') \right\} \left[\sum_{n=2}^{\infty} \frac{2i\alpha_n r' \frac{d}{dr'} J_1(\eta_n r') J_1(\eta_n r)/r}{\omega\mu_o \left[(\eta_n r_o)^2 - 1\right] J_1^2(\eta_n r_o)} - \right.$$

$$- 2i\omega\epsilon_o \epsilon_1 \sum_{\ell=1}^{\infty} \frac{J_1(\lambda_\ell r') \frac{d}{dr} J_1(\lambda_\ell r)}{(\lambda_\ell r_o)^2 \gamma_\ell J_o^2(\lambda_\ell r_o)} + \tag{35}$$

$$+ \int_0^\infty \frac{d\zeta\xi}{\omega\mu_o\zeta} r' \frac{d}{dr'} J_1(\zeta r') J_1(\zeta r)/r + \omega\epsilon_o\epsilon_2 \int_0^\infty \frac{d\zeta}{\zeta\xi} J_1(\zeta r') \frac{d}{dr} J_1(\zeta r) \Bigg] .$$

Multiply (27) and (28) by $r\mathcal{E}_2(r)dr$ and $r\mathcal{E}_1(r)dr$ respectively and integrate both sides from 0 to r_0. Then, adding the results, the quantity Λ which is related to the aperture admittance, Y(0) can be expressed in the following manner:

$$\Lambda = \frac{1}{L_o^2} \Bigg[\int_0^{r_o} dr\ r\ \mathcal{E}_1(r)\ L_3(r,r')\ \mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_2(r)\ L_2(r,r')\ \mathcal{E}_2(r') +$$

$$+ \int_0^{r_o} dr\ r\ \mathcal{E}_2(r)\ L_1(r,r')\ \mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_1(r)\ L_4(r,r')\ \mathcal{E}_2(r') \Bigg] . \tag{36}$$

It is shown in Appendix B, that Λ (and hence, Y(0) given by (36) is stationary with respect to the first-order simultaneous variation of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ about their respective correct values.

Note that one could have obtained an alternative variational expression for Λ in terms of the aperture magnetic fields $\mathcal{H}_1(r)$ and $\mathcal{H}_2(r)$ instead of the aperture electric fields $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ as shown in Equation (36). However, it will not be attempted here.

5. *Application of the variational principles*

From the definitions of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ given by (17) and the expressions (18) and (19), it implies that the aperture electric fields $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ can be expressed in the following manner;

$$\mathcal{E}_1(r) = \sum_{n=1}^{\infty} a_n \frac{J_1(\eta_n r)}{r} + \sum_{\ell=1}^{\infty} b_\ell\ \lambda_\ell J_1'(\lambda_\ell r), \tag{37}$$

and

$$\mathcal{E}_2(r) = \sum_{n=1}^{\infty} a_n \eta_n\ J_1'(\eta_n r) + \sum_{\ell=1}^{\infty} b_\ell \frac{J_1(\lambda_\ell r)}{r}, \tag{38}$$

where a_1 is related to A_0 and Γ, whereas $a_n (n \geq 2)$ and b_ℓ corespond to A_n $(n \geq 2)$ and B_ℓ respectively.

Now, substituting (37) and (38) for $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ respectively into the variational expression (36), Λ can be expressed in terms of a_n and b_ℓ in the following way (using the integrals in Appendix A) after some suitable manipulations:

$$\Lambda = - \sum_{\ell=1}^{\infty} c^{(1)} \hat{b}_\ell^{\,2} - \sum_{n=2}^{\infty} c_n^{(2)} \hat{a}_n^{\,2} + \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \hat{a}_n \hat{a}_{n'} \left\{ c_{nn'}^{(3)} + c_{nn'}^{(5)} \right\} +$$

$$+ \sum_{\ell=1}^{\infty} \sum_{\ell'=1}^{\infty} \hat{b}_\ell \hat{b}_{\ell'}\ c_{\ell\ell'}^{(4)} + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \hat{a}_n \hat{b}_\ell\ c_{n\ell}^{(6)} \tag{39}$$

The symbols introduced in the expression (39) are defined as follows:

$$c_{\ell}^{(1)} = \frac{2i\omega\epsilon_0\epsilon_1(\lambda\, r_o)^2\, J_o^{\,2}(\lambda\, r_o)}{\gamma_\ell\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,4}(\eta_1 r_o)},\ \ell \geq 1, \tag{40}$$

$$c_n^{(2)} = \frac{-2i\alpha_n\left[(\eta_n r_o)^2 - 1\right] J_1^{\,2}(\eta_n r_o)}{\omega\mu_o\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,2}(\eta_1 r_o)},\ n \geq 2, \tag{41}$$

$$c_{nn'}^{(3)} = \frac{4\, r_o^{\,2}\eta_n^{\,2}\eta_{n'}^{\,2}\, J_1(\eta_n r_o) J_1(\eta_{n'} r_o)}{\omega\mu_o\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,4}(\eta_1 r_o)} \int_0^\infty \frac{d\zeta\ \xi\ \zeta\ J_1'^{\,2}(\zeta r_o)}{(\eta_n^{\,2}-\zeta^2)\ (\eta_{n'}^{\,2}-\zeta^2)} = c_{n'n}^{(3)}, \tag{42}$$

$$n, n' \geq 1,$$

$$c_{\ell\ell'}^{(4)} = \frac{4\omega\epsilon_o\epsilon_2\lambda_\ell r_o\lambda_{\ell'} r_o J_o(\lambda_\ell r_o) J_o(\lambda_{\ell'} r_o)}{\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,4}(\eta_1 r_o)}\ \times$$

$$\times \int_0^\infty \frac{d\zeta\ \zeta^3 J_1^{\,2}(\zeta r_o)}{\xi\ (\zeta^2 - \lambda_\ell^{\,2})\ (\zeta^2 - \lambda_{\ell'}^{\,2})} = c_{\ell'\ell}^{(4)}\ ;\ell, \ell' \geq 1, \tag{43}$$

$$c_{nn'}^{(5)} = \frac{4\omega\epsilon_o\epsilon_2\ J_1(\eta_n r_o) J_1(\eta_{n'} r_o)}{\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,4}(\eta_1 r_o)} \int_0^\infty \frac{d\zeta\ J_1^{\,2}(\zeta r_o)}{\zeta\ \xi} = c_{n'n}^{(5)}, \tag{44}$$

$$n, n' \geq 1,$$

$$c_{n\ell}^{(6)} = \frac{8\omega\epsilon_o\epsilon_2\ J_1(\eta_n r_o)\ (\lambda_\ell r_o)\ J_o(\lambda_\ell r_o)}{\left[(\eta_1 r_o)^2 - 1\right]^2 J_1^{\,4}(\eta_1 r_o)} \int_0^\infty \frac{d\zeta\ \zeta\ J_1^{\,2}(\zeta r_o)}{\xi\,(\zeta^2 - \lambda_\ell^{\,2})} \neq c_{\ell n}^{(6)}, \tag{45}$$

$$n, \ell \geq 1,$$

$$\left.\begin{aligned} \hat{a}_m &= a_m/a_1, \text{ where } m = n \text{ or } n' \\ \hat{b}_j &= b_j/a_1, \text{ where } j = \ell \text{ or } \ell' \end{aligned}\right\} \tag{46}$$

consequently $\hat{a}_1 = 1$

The expressions $\sum_{n=1}\sum_{n'=1} \hat{a}_n\hat{a}_{n'}\left\{c_{nn'}^{(3)} + c_{nn'}^{(5)}\right\}$ and $\sum_{\ell=1}\sum_{n=1} \hat{a}_n\hat{b}_\ell c_{n\ell}^{(6)}$

can be rewritten as:

$$\sum_{n=1}\sum_{n'=1} \hat{a}_n\hat{a}_{n'}\left\{c_{nn'}^{(3)} + c_{nn'}^{(5)}\right\} = \left\{c_{11}^{(3)} + c_{11}^{(5)}\right\} + 2\sum_{n=2}\hat{a}_n\left\{c_{n1}^{(3)} + c_{n1}^{(5)}\right\} +$$

$$+ \sum_{n=2}\sum_{n'=2}\hat{a}_n\hat{a}_{n'}\left\{c_{nn'}^{(3)} + c_{nn'}^{(5)}\right\}, \tag{47i}$$

$$\sum_{\ell=1}\sum_{n=1}\hat{a}_n\hat{b}_\ell\ c_{n\ell}^{(6)} = \sum_{\ell=1}\hat{b}_\ell\ c_{1\ell}^{(6)} + \sum_{\ell=1}\sum_{n=2}\hat{a}_n\hat{b}_\ell\ c_{n\ell}^{(6)}\ . \tag{47ii}$$

Now, using (47i) and (47ii), the expression for Λ given by (39) can be rewritten in the following manner;

$$\Lambda = \left\{ c_{11}^{(3)} + c_{11}^{(5)} \right\} + 2 \sum_{n=2} \hat{a}_n \left\{ c_{n1}^{(3)} + c_{n1}^{(5)} \right\} + \sum_{n=2} \sum_{n'=2} \hat{a}_n \hat{a}_{n'} \left\{ c_{nn'}^{(3)} + c_{nn'}^{(5)} \right\} +$$
$$+ \sum_{\ell=1} \sum_{\ell'=1} \hat{b}_\ell \hat{b}_{\ell'} c_{\ell\ell'}^{(4)} + \sum_{\ell=1} \sum_{n=2} \hat{a}_n \hat{b}_\ell c_{n\ell}^{(6)} + \sum_{\ell=1} \hat{b}_\ell c_{1\ell}^{(6)} -$$
$$- \sum_{\ell=1} \hat{b}_\ell^{\,2} c_\ell^{(1)} - \sum_{n=2} \hat{a}_n^{\,2} c_n^{(2)}. \tag{48}$$

Owing to the stationary character of Λ, on differentiating (48) with respect to $\hat{a}_n$ and $\hat{b}_\ell$, the following infinite set of simultaneous equations are obtained;

Since $\hat{a}_n$ and $\hat{b}_\ell$ are independent, one then has;

$$\frac{\partial \Lambda}{\partial \hat{a}_n} = 2 \left\{ c_{n1}^{(3)} + c_{n1}^{(5)} \right\} + 2 \sum_{n'=2} \hat{a}_{n'} \left\{ c_{nn'}^{(3)} + c_{nn'}^{(5)} \right\} -$$
$$-2 \hat{a}_n c_n^{(2)} + \sum_{\ell=1} \hat{b}_\ell c_{n\ell}^{(6)} = 0, \tag{49i}$$
$$n = 2, 3 \ldots,$$

and

$$\frac{\partial \Lambda}{\partial \hat{b}_\ell} = -2\hat{b}_\ell c_\ell^{(1)} + 2 \sum_{\ell'=1} \hat{b}_{\ell'} c_{\ell\ell'}^{(4)} + c_{1\ell}^{(6)} + \sum_{n=2} \hat{a}_n c_{n\ell}^{(6)} = 0, \tag{49ii}$$
$$\ell = 1, 2, 3, \ldots$$

One can also arrive at the results (49i) and (49ii) directly without resorting the variational properties of Λ with respect to the independent variation of $\hat{a}_n$ and $\hat{b}_\ell$ respectively in the following manner.

Multiply (27) and (28) by $r\, d/dr_1 J\, (\eta_n r)\, dr$ and $J_1(\eta_n r)\, dr$ respectively (where $n \geq 2$), and then integrate from 0 to r_o using the relations (37) and (38). When these two resulting expressions are added, one obtains (49i) with the help of the results in Appendix A and the relations (40) to (46). In order to obtain (49ii), multiply (27) and (28) by $J_1(\lambda_\ell r)\, dr$ and $r\, d/dr\, J_1(\lambda_\ell r)\, dr$ respectively and then integrate from 0 to r_o using the relation (37) and (38) followed by the same procedure sought for the derivation of (49i) by this direct method.

Now, multiplying (49i) by $\hat{a}_n$ and then summing over $n = 2$ to ∞, we have:

$$2 \sum_{n=2} \hat{a}_n \left\{ c_{n1}^{(3)} + c_{n1}^{(5)} \right\} + 2 \sum_{n=2} \sum_{n'=2} \hat{a}_n \hat{a}_{n'} \left\{ c_{nn'}^{(3)} + c_{nn'}^{(5)} \right\} -$$
$$-2 \sum_{n=2} \hat{a}_n^{\,2} c_n^{(2)} + \sum_{n=2} \sum_{\ell=1} \hat{a}_n \hat{b}_\ell c_{n\ell}^{(6)} = 0.$$

Similarly, multiplying (49ii) by $\hat{b}_\ell$ and summing over $\ell = 1$ to ∞, we have:

$$- \sum_{\ell=1} \hat{b}_\ell^{(2)} c_\ell^{(1)} + \sum_{\ell=1} \sum_{\ell'=1} \hat{b}_\ell \hat{b}_\ell c_{\ell\ell'}^{(4)} + \sum_{\ell=1} \hat{b}_\ell c_{1\ell}^{(6)} +$$
$$+ \sum_{\ell=1} \sum_{n=2} \hat{a}_n \hat{b}_\ell c_{n\ell}^{(6)} = 0 \tag{51}$$

Now, in view of the relations (50) and (51), the expression for Λ given by (48) can be represented in the following fashion:

$$\Lambda = \left\{ c_{11}^{(3)} + c_{11}^{(5)} \right\} + \sum_{\ell=1}^{\infty} \hat{b}_\ell^{\,2}\, c_\ell^{(1)} - \sum_{\ell=1}^{\infty}\sum_{\ell'=1}^{\infty} \hat{b}_\ell \hat{b}_{\ell'}\, c_{\ell\ell'}^{(4)} + \tag{52}$$
$$+ \sum_{n=2}^{\infty} \hat{a}_n^{\,2}\, c_n^{(2)} - \sum_{n=2} \sum_{n'=2} \hat{a}_n \hat{a}_{n'} \left\{ c_{nn'}^{(3)} + c_{nn'}^{(5)} \right\} -$$
$$- \sum_{\ell=1}^{\infty}\sum_{n=2}^{\infty} \hat{a}_n \hat{b}_\ell\, c_{n\ell}^{(6)} .$$

Note that the equation (49) provides the relationship among the coefficients. Therefore, for numerical computation of Λ, both (49) and (52) should be considered simultaneously.

6. *Iteration method -- an alternative approach*

In order to follow this approach of solving the set of integral equations for the unknown coefficients Γ, A_n, B_1, $T_1(\zeta)$ and $T_2(\zeta)$ given by the Equations (18) to (21), we obtain first a set of relations analogous to (24) to (26) (which were obtained from (18) and (19) using this time (20) and (21). To do this, let us multiply (20) and (21) by $r\, d/dr\, J_1(\eta_n r)\, dr$ and $J(\eta_n r)\, dr$ (where $n = 1, 2, 3, \ldots$) respectively and then integrating both sides from 0 to r_0, we find after adding (using the results of Appendix A):

$$A_0(1-\Gamma) = \frac{2\,\eta_1^2}{\beta_1\left[(\eta_1 r_o)^2 - 1\right] J_1(\eta_1 r_o)} \left[\eta_1^2 r_o \int_0^\infty \frac{d\zeta\; \zeta^2\, \xi\, J_1'(\zeta r_o) T_2(\zeta)}{\eta_1^2 - \zeta^2} - \right.$$
$$\left. - \omega\, \epsilon_o \epsilon_2 \int_0^\infty d\zeta\; \zeta\, J_1(\zeta r_o)\, T_1(\zeta) \right] , \tag{53}$$

$$A_n = \frac{-2i\,\eta_n^2}{\alpha_n\left[(\eta_n r_o)^2 - 1\right] J_1(\eta_n r_o)} \left[\eta_n^2 r_o \int_0^\infty \frac{d\zeta\; \zeta^2\, \xi\, J_1'(\zeta r_o)\, T_2(\zeta)}{\eta_n^2 - \zeta^2} = \right.$$
$$\left. -\omega \epsilon_o \epsilon_2 \int_0^\infty d\zeta\; \zeta\; J_1(\zeta r_o)\; T_1(\zeta) \right] , \quad n \geqq 2. \tag{54}$$

Once again, multiply (20) and (21) by $J_1(\lambda_\ell r)\, dr$ and $r\, d/dr\, J_1(\lambda_\ell r)\, dr$ respectively and integrate both sides from 0 to r_0. Then adding the resulting expressions, we get with the aid of the results of Appendix A:

$$B_\ell = \frac{2\,\lambda_\ell\, \epsilon_2}{r_o \epsilon_1\, J_o(\lambda_\ell r_o)} \int_0^\infty d\zeta\; \zeta^3\, \frac{J_1(\zeta r_o)\; T_1(\zeta)}{\zeta^2 - \lambda_\ell^2} , \quad \ell \geqq 1. \tag{55}$$

Though the relation (55) is derived from the requirement of the continuity of the tangential components of the magnetic fields across the circular aperture, one could have obtained it also from the continuity of the axial component of the electric displacement vector (i.e., $\epsilon_1 E_{z1} = \epsilon_2 E_{z2}$ at $z = 0$) across the aperture. This second interpretation of the result (55) will be found useful in obtaining approximate results.

Now, the expressions (22), (23) and (53) to (55), excluding the portions containing $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$, constitute the required equations to be solved for Γ, A_n, B , $T_1(\zeta)$ and $T_2(\zeta)$ by iteration method. This iteration method of solving the above mentioned simultaneous integral equations consists of successive approximations which need an initial approximation or a trial solution. Then a systematic successive use of the equations to be solved yields better and better approximations to the solution sought for. The more accurate the initial or trial solution is, the lesser the number of iterations necessary to obtain a required accuracy. The main idea behind this procedure is the same as that one adopts in obtaining a more accurate result of a quantity from its variational expression. Since a physical insight into a problem helps often to select judicially, a better initial and approximate trial solution, let us digress for a moment to discuss the relative importance of the various modes excited by the discontinuity of the open end (circular aperture) of the waveguide.

If it is assumed that the waveguide is circularly symmetric and there is no discontinuity in the boundary along the angular (ϕ) direction, an incident TE_{11} mode will excite TE_{1n} and $TM_{1\ell}$ modes ($n, \ell \geq 1$) inside the waveguide, where n and ℓ correspond to various radial modes. Also, if the circular waveguide is designed in such a way that it allows an unattenuated propagation only for TE_{11} mode, all other higher-order modes TE_{1n} ($n \geq 2$) and $TM_{1\ell}$ ($\ell \geq 1$) will be attenuated. The higher the cut-off wave number of a mode is, the higher will be the attenuation of that mode. In view of the relations (5ii), (15ii) and (55), one can argue that a situation may arise for which B_ℓ and hence, the right-hand side of the integral (55) are vanishingly small; however, $T_1(\zeta)$ is finite and not necessarily negligible. This implies that $TM_{1\ell}$ modes inside the waveguide and consequently, the axial electric field at the circular aperture are negligibly small. It may be noted, therefore, that though the axial electric field in the vicinity of the aperture on the free-space side (i.e., unbounded space) may be vanishingly small, it may not be neglected (since $T_1(\zeta)$ is not negligible) everywhere else. Another important point which deserve attention is that though the cut-off wave number of the dominant TE_{11} mode (for which $\eta_1 r_o = 1.841$) is smaller than that of the TM_{11} mode for which $\lambda_1 r_o = 3.832$, the cut-off wave number of the TE_{12} mode, given by $\eta_2 r_o = 5.331$ is not smaller. Consequently, if TM_{11} mode is negligible, so also is the TE_{12} mode. The converse is not necessarily true. The preceding discussions, therefore, suggest that the judicial choice of the first (or initial) and the next higher approximations to the solution should be made in the following manner. In the first approximation, both $A_n (n \geq 2)$ and B_ℓ ($\ell \geq 1$) are negligible but neither of $T_1(\zeta)$ and $T_2(\zeta)$ can be disregarded. This means that, in this approximation only, the TE_{11} mode exists in the waveguide, however, both TE and TM mode are excited in the unbounded space.

In the next higher approximation, all $A_n (n \geq 2)$ and B_ℓ ($\ell \geq 2$) are vanishingly small nut not B_1. In this situation only, TE_{11} and TM_{11} modes contribute to the fields. For the third higher approximation, $A_2 \neq 0$, $B_1 \neq 0$, but all other A_n ($n \geq 3$) and B_ℓ ($\ell \geq 2$) are negligible. In this manner, the rest of the higher-ordered successive approximations to the solution, with any desired degree of accuracy, can be achieved in principle.

Let us now turn to the procedure of obtaining the successive approximations to the solutions of the unknown quantities appearing in (22), (23), and (53) to (55). Using the superscript s on Γ, A_n, B_ℓ, $T_1(\zeta)$ and $T_2(\zeta)$ to designate their respective s^{th} approximation (where $s = 1, 2, \ldots$), the argument made in the preceding paragraph motivates the following representations of the unknown quantities sought for:

$$\Gamma = \sum_{s=1}^{\infty} \Gamma^{(s)} \tag{56i}$$

$$A_n = \sum_{s=1}^{\infty} A_n^{(s+2)} \tag{56ii}$$

$$B_\ell = \sum_{s=1}^{\infty} B_\ell^{(s+1)} \tag{56iii}$$

$$T_1 = \sum_{s=1}^{\infty} T_1^{(s)} \tag{56iv}$$

$$T_2 = \sum_{s=1}^{\infty} T_2^{(s)} \tag{56v}$$

Therefore, it should be understood in the above representations that the lowest non-vanishing terms of A_n and B_ℓ are $A_2^{(3)}$ and $B_1^{(2)}$ respectively. Now, the relations (22), (23) and (53) to (55) can be re-expressed in terms of $\Gamma^{(s)}$, $A_n^{(s+2)}$, $B_\ell^{(s+1)}$, $T_1^{(s)}$ and $T_2^{(s)}$ in the following manner:

$$T_1^{(s)} = \frac{1}{\zeta^2 \xi}\left[- \omega\mu_o A_o \frac{(1+\Gamma^{(s)})}{\eta_n^2} J_1(\eta_1 r_o) + \omega\mu_o \sum_{n=2}^{\infty} \frac{A_n^{(s+2)}}{\eta_n^2} J_1(\eta_n r_o) - i\sum_{\ell=1}^{\infty} \frac{B_\ell^{(s+1)} \gamma_\ell r_o \zeta^2 J_o(\lambda_\ell r_o)}{\lambda_\ell(\zeta^2 - \lambda_\ell^2)}\right] J_1(\zeta r_o), \tag{57}$$

$$T_2^{(s)} = \frac{r_o}{\zeta}\left[\frac{A_o(1+\Gamma^{(s)})J_1(\eta_1 r_o)}{(\eta_1^2 - \zeta^2)} - \sum_{n=2}^{\infty} \frac{A_n^{(s+2)} J_1(\eta_n r_o)}{(\eta_n^2 - \zeta^2)}\right] J_1'(\zeta r_o), \tag{58}$$

$$A_o(1-\Gamma^{(s)}) = \frac{2\eta_1^2}{\beta_1\left[(\eta_1 r_o)^2 - 1\right] J_1(\eta_1 r_o)}\left[\eta_1^2 r_o \int_0^{\infty} \frac{d\zeta\, \zeta^2\, \xi\, J_1'(\zeta r_o)}{\eta_1^2 - \zeta^2} T_2^{(s)}(\zeta) - \omega\epsilon_o\epsilon_2 \int_0^{\infty} d\zeta\, \zeta\, J_1(\zeta r_o)\, T_1^{(s)}(\zeta)\right], \tag{59}$$

$$A_n^{(s+2)} = -\frac{2i\,\eta_n}{\alpha_n\left[(\eta_n r_o)^2 - 1\right] J_1(\eta_n r_o)}\left[\eta_n^2 r_o \int_0^{\infty} \frac{d\zeta\, \zeta^2\, \xi\, J_1'(\zeta r_o)}{\eta_n^2 - \zeta^2} T_2^{(s)}(\zeta) - \omega\epsilon_o\epsilon_2 \int_0^{\infty} d\zeta\, \zeta\, J_1(\zeta r_o)\, T_1^{(s)}(\zeta)\right], \tag{60}$$

$$B_\ell^{(s+1)} = \frac{2\lambda_\ell \epsilon_2}{r_o \epsilon_1 J_o(\lambda_\ell r_o)} \int_0^{\infty} \frac{d\zeta\, \zeta^3 J_1(\zeta r_o)\, T_1^{(s)}(\zeta)}{\zeta^2 - \lambda_\ell^2}\ . \tag{61}$$

Let us now consider the second-order approximate solution for which $s = 1$, and $A_n^{(3)}$ is negligible but not $B_1^{(2)}$. Then, we have;

$$T_2^{(1)} = \frac{A_o r_o J_1(\eta_1 r_o)\, J_1'(\zeta r_o)}{\zeta(\eta_1^2 - \zeta^2)}\left\{1 + \Gamma^{(1)}\right\}, \tag{62}$$

$$T_1^{(1)} = -\frac{1}{\zeta^2 \xi}\left[\frac{\omega\mu_o A_o J_1(\eta_1 r_o)}{\eta_1^2}\left\{1+\Gamma^{(1)}\right\} + i\,\frac{B_1^2\, \gamma_1 r_o \zeta\, J_o(\lambda_1 r_o)}{\lambda_1(\zeta^2 - \lambda_1^2)}\right] J_1(\zeta r_o)\ , \tag{63}$$

$$\frac{B_1^{(2)}}{A_o\left[1+\Gamma^{(1)}\right]} = \frac{2\omega\mu_o\epsilon_2\lambda_1 J_1(\eta_1 r_o)}{\epsilon_1 r_o \eta_1^2 J_o(\lambda_1 r_o)}\left[\frac{\displaystyle\int_0^\infty \frac{d\zeta\ \zeta\ J_1^2(\zeta r_o)}{\xi\,(\lambda_1^2-\zeta^2)}}{1+\dfrac{2i\epsilon_2\gamma_1}{\epsilon_1}\displaystyle\int_0^\infty \frac{d\zeta\ \zeta^3 J_1^2(\zeta r_o)}{\xi\,(\lambda_1^2-\zeta^2)^2}}\right]. \tag{64}$$

The relation (59) should also be considered for s = 1, together (62) to (64) for the derivation of the aperture admittance, which can be shown to be as;

$$Y(0) = \frac{\beta_1}{\omega\mu_o}\left[\frac{1-\Gamma^{(1)}}{1+\Gamma^{(1)}}\right] = \frac{2\,\eta_1^2(\eta_1 r_o)^2}{\omega\mu_o\left[(\eta_1 r_o)^2-1\right]}\int_0^\infty \frac{d\zeta\ \zeta\ \xi\ J_1'^2(\zeta r_o)}{(\eta_1^2-\zeta^2)^2} +$$

$$+\frac{2\omega\epsilon_o\epsilon_2}{\left[(\eta_1 r_o)^2-1\right]}\int_0^\infty \frac{d\zeta\ J_1^2(\zeta r_o)}{\zeta\ \xi} - \frac{i4\omega\epsilon_o\epsilon_2^2\gamma_1}{\epsilon_1\left[(\eta_1 r_o^2)-1\right]}\ \mathrm{x}$$

$$\mathrm{x}\left[\frac{\left\{\displaystyle\int_0^\infty \frac{d\zeta\ \zeta\ J_1^2(\zeta r_o)}{\xi(\lambda_1^2-\zeta^2)}\right\}^2}{1+\dfrac{2i\epsilon_2\gamma_1}{\epsilon_1}\displaystyle\int_0^\infty \frac{d\zeta\ \zeta^3 J_1^2(\zeta r_o)}{\xi(\zeta^2-\lambda_1^2)^2}}\right].$$

The equation (65) determines $\Gamma^{(1)}$. Thus, when $\Gamma^{(1)}$ is known, the remaining unknown quantities can be calculated from (62) to (64). Note that for the lowest-order approximation, for which $B_1^{(2)}$ is negligible, the last term on the right-hand side of (65) becomes also vanishingly small, for which case, numerical results have been found recently by Bailey, Samaddar and Swift.

7. Approximate solution derived from the variational expression

In this section, it will be demonstrated that the approximate solution obtained in the preceding section dealing with successive iteration method can also be obtained from the variational result given by (52) (subject to the relation (49)) for the modified aperture admittance Λ.

Since the lowest value of n or n' in (49) and (52) is 2, any term having the subscript n or n' in those equations is negligible for the same degree of approximate solution obtained in the Section 6. It should be noted, however, that only the lowest value of ℓ (or ℓ'), namely ℓ, ℓ' = 1, will contribute to this approximation.

With the aid of these assumptions, one finds the following approximate result from (49) and (52) respectively.

$$-2\ \hat{b}_1 c_1^{(1)} + 2\ \hat{b}_1 c_{11}^{(4)} + c_{11}^{(6)} = 0 \tag{66}$$

and

$$\Lambda = c_{11}^{(3)} + c_{11}^{(5)} + \hat{b}_1^2\left\{c_1^{(1)} - c_{11}^{(4)}\right\} \tag{67}$$

It may be recalled here that for the lowest-order approximation for which the contribution from the TM_{11} mode can be disregarded, the quantities $\hat{b}_1$, $c_{11}^{(6)}$, $c_1^{(1)}$ and $c_{11}^{(4)}$ appaering in (66) and (67) are also negligible. This implies that the axial component of the electric field at the circular aperture is vanishingly small.

The substitution for $\hat{b}_1$ from (66) leads Equation (67) to:

$$\Lambda = c_{11}^{(3)} + c_{11}^{(5)} + \frac{\left[c_{11}^{(6)}\right]^2}{4\left[c_1^{(1)} - c_{11}^{(4)}\right]}, \tag{68}$$

or

$$Y(0) = \frac{\beta_1}{\omega\mu_o}\left[\frac{1-\Gamma}{1+\Gamma}\right] = \frac{\left[(\eta_1 r_o)^2 - 1\right] J_1^2(\eta_1 r_o)}{2}\left[c_{11}^{(3)} + c_{11}^{(5)} + \frac{\left[c_{11}^{(6)}\right]^2}{4\left[c_1^{(1)} - c_{11}^{(4)}\right]}\right]. \tag{69}$$

In a straightforward manner, it can now be shown (with the aid of the relations (40) and (42) to (45) that the first, second and the third term of the right-hand side expression of (69) are identical to the corresponding terms respectively of the right-hand side of (65). Thus, it is demonstrated here that the approximate solutions derived from two different methods discussed in this paper agree with one another.

An inspection of the expressions (39), (65) and (69) which are independent of A_0, σ_1 and σ_2, shows that the aperture admittance depends neither on the amplitude nor on the type of polarization (linear or elliptic) of the incident TE_{11} mode. In general, the aperture admittance, Y(0), is a complex quantity. For a proper matching of the aperture antenna to the unbounded medium, it is desirable that the imaginary part of Y(0) should approach zero, while the real part of Y(0) tends to $\beta_1/\omega\mu_o$ (which is also equivalent to Γ approaching zero).

8. *Radiation Fields*

In view of the representations (14i) and (14ii), the expressions (11) to (13) provide the asymptotic values (far fields) of the Hertz potentials $F_1(r,\phi,z)$ and $F_2(r,\phi,z)$. From these Hertz potentials, all the cylindrical components of the radiated electromagnetic fields can easily be calculated with the aid of the relations in (4). As an illustration, we simply present here the z-components of the far field:

$$E_{z2} \sim -\Phi_1(\phi)\, K_2^3\, T_1(\zeta = K_2 \sin\theta)\, \sin^2\theta \cos\theta\, \frac{e^{iK_2R}}{R}, \tag{70}$$

$$H_{z2} \sim -\Phi_2(\phi)\, K_2^3\, T\,(\zeta = K_2 \sin\theta)\, \sin^2\theta \cos\theta\, \frac{e^{iK_2R}}{R} \tag{71}$$

where,

$$\left.\begin{aligned} z &= R\cos\theta \\ r &= R\sin\theta \\ K_2^2 &= \omega^2\mu_o\epsilon_o\epsilon_2 \end{aligned}\right\}. \tag{72}$$

Now, instead of expressing the remaining cylindrical components of the far field, it may be desirable for practical purposes, to know the spherical components (θ and ϕ components which vary as $\exp(iK_2R)/R$). These spherical components of the radiated field can be shown to possess the following representations:

$$E_{\theta 2} \sim \frac{\Phi_1(\phi)}{2} K_2^{\,3} T_1(\zeta = K_2 \sin\theta) \sin 2\theta . \frac{e^{iK_2R}}{R}, \tag{73}$$

$$E_{\phi 2} \sim -\frac{\Phi_2(\phi)}{2} \omega\mu_o K_2^{\,2} T_2(\zeta = K_2 \sin\theta) \sin 2\theta . \frac{e^{iK_2R}}{R}, \tag{74}$$

$$H_{\theta 2} \sim -\sqrt{\epsilon_o \epsilon_2/\mu_o}\, E_{\phi 2}, \tag{75}$$

$$H_{\phi 2} \sim -\sqrt{\epsilon_o \epsilon_2/\mu_o}\, E_{\theta 2}. \tag{76}$$

Note that the ϕ-component of a field vector is the same in both the cylindrical and spherical coordinate systems.

For the final calculation of $T_1(\zeta)$ and $T_2(\zeta)$, first of all Γ(or $Y(0)$ or Λ) must be determined either from the relation (52) or by the method discussed in Section 6 (see, for example, Equation (65)). Then, b_ℓ and A_n (depending on the degree of approximations desired) are to be evaluated. Finally, $T_1(\zeta)$ and $T_2(\zeta)$ can be expressed in terms of Γ, A_n and B_ℓ (see, for example, Equations (22) and (23)). The formal expressions of the far fields given by (70) to (76) are valid for any order of successive approximations subject to their asymptotic behavior at large distances.

APPENDIX A

SOME INTEGRALS INVOLVING BESSEL FUNCTIONS

The following integrals have been used frequently in the text:

$$I_1 = \int_0^{r_o} dr \left[J_1(\zeta r) \frac{d}{dr} J_1(\eta_n r) + J_1(\eta_n r) \frac{d}{dr} J_1(\zeta r) \right], \quad n = 1, 2, \ldots \tag{A-1}$$

$$I_2 = \int_0^{r_o} dr \left[r \frac{d}{dr} J_1(\lambda_\ell r) \frac{d}{dr} J_1(\zeta r) + \frac{J_1(\lambda_\ell r)\, J_1(\zeta r)}{r} \right], \quad \ell = 1, 2, \ldots \tag{A-2}$$

$$I_3 = \int_0^{\infty} dr \left[J_1(\zeta r) \frac{d}{dr} J_1(\zeta' r) + J_1(\zeta' r) \frac{d}{dr} J_1(\zeta r) \right] \tag{A-3}$$

$$I_4 = \int_0^{\infty} dr \left[r \frac{d}{dr} J_1(\zeta r) \frac{d}{dr} J_1(\zeta' r) + \frac{J_1(\zeta r)\, J_1(\zeta' r)}{r} \right] \tag{A-4}$$

$$I_5 = \int_0^{r_o} dr \left[r \frac{d}{dr} J_1(\eta\ r) \frac{d}{dr} J_1(\eta_m r) + \frac{J_1(\eta_n r)\, J_1(\eta_m r)}{r} \right] \tag{A-5}$$

$$m, n = 1, 2, 3, \ldots$$

$$I_6 = \int_0^{r_o} dr \left[J_1(\lambda_\ell r) \frac{d}{dr} J_1(\eta_n r) + J_1(\eta_n r) \frac{d}{dr} J_1(\lambda_\ell r) \right] \tag{A-6}$$

$$I_7 = \int_0^{r_o} dr \left[r \frac{d}{dr} J_1(\eta_n r) \frac{d}{dr} J_1(\zeta r) + \frac{J_1(\eta_n r)\, J_1(\zeta r)}{r} \right] \tag{A-7}$$

$$I_8 = \int_0^{r_o} dr \left[r \frac{d}{dr} J_1(\lambda_\ell r) \frac{d}{dr} J_1(\lambda_{\ell'} r) + \frac{J_1(\lambda_\ell r)\, J_1(\lambda_{\ell'} r)}{r} \right], \quad \ell, \ell' = 1, 2, \ldots \tag{A-8}$$

$$I_9 = \int_0^{r_o} dr \left[J_1(\lambda_\ell r) \frac{d}{dr} J_1(\zeta r) + J_1(\zeta r) \frac{d}{dr} J_1(\lambda_\ell r) \right] \tag{A-9}$$

Now, using the differential equation satisfied by $J_1(z)$, [Watson, (1944)] the properties of Hankel transform [Morse and Feshbach, 1953] and the relations $J_1'(\eta_n r) = 0$, $J_1(\lambda_\ell r) = 0$, the values of the above integrals can readily be shown as presented in the following:

$$I_1 = J_1(\eta_n r_o)\, J_1(\zeta r_o) \tag{A-10}$$

$$I_2 = \frac{\lambda_\ell r_o\, \zeta^2 J_1(\zeta r_o)\, J_0(\lambda_\ell r_o)}{\zeta^2 - \lambda_\ell^2} \tag{A-11}$$

$$I_3 = 0 = I_9 = I_6 = 0 \tag{A-12}$$

$$I_4 = \zeta\, \delta(\zeta - \zeta') \tag{A-13}$$

$$I_5 = \frac{\left[(\eta_m r_o)^2 - 1\right]}{2} J_1^2(\eta_m r_o)\, \delta_{mn} \tag{A-14}$$

$$I_7 = \frac{\eta_n^2 r_o\, \zeta\, J_1(\eta_n r_o)\, J_1'(\zeta r_o)}{(\eta_n^2 - \zeta^2)} \tag{A-15}$$

$$I_8 = -\frac{(\lambda_\ell r_o)^2}{2} J_0^2(\lambda_\ell r_o)\, \delta_{\ell\ell'} \tag{A-16}$$

The quantities δ_{mn} and $\delta_{\ell\ell'}$ are Kronecker delta functions, whereas $\delta(\zeta - \zeta')$ is a Dirac delta function.

APPENDIX B

PROOF OF THE STATIONARY CHARACTER OF Λ

In order to prove that Λ given by the relation (36) is in its variational form with respect to the first-order simultaneous variation of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$, it will be found useful to establish some sort of symmetry properties of the operators $L_s(r, r')$, where s = 1, 2, 3 and 4. Defining $L_s(r', r)$ as the operator, which is obtained by interchanging r and r' in $L_s(r, r')$, an inspection of the expressions (32) to (35) shows that the following relations hold true:

$$\int_0^{r_o} dr\, r\, \mathcal{E}_2(r)\, L_2(r, r')\, \mathcal{E}_2(r') = \int_0^{r_o} dr'\, r'\, \mathcal{E}_2(r')\, L_2(r', r)\, \mathcal{E}_2(r) \tag{B-1}$$

$$\int_0^{r_o} dr\ r\ \mathcal{E}_1(r)\ L_3(r,r')\mathcal{E}_1(r') = \int_0^{r_o} dr'\ r'\ \mathcal{E}_1(r')L_3(r',r)\mathcal{E}_1(r) \qquad \text{(B-2)}$$

Though

$$\int_0^{r_o} dr\ r\ \mathcal{E}_2(r)\ L_1(r,r')\mathcal{E}_1(r') \neq \int_0^{r_o} dr'\ r'\ \mathcal{E}_2(r')L_1(r',r)\mathcal{E}_1(r) \qquad \text{(B-3)}$$

$$\int_0^{r_o} dr\ r\ \mathcal{E}_1(r)\ L_4(r,r')\ \mathcal{E}_2(r') \neq \int_0^{r_o} dr'\ r'\ \mathcal{E}_1(r')L_4(r',r)\mathcal{E}_2(r) \qquad \text{(B-4)}$$

we have

$$\int_0^{r_o} dr\ r\ \mathcal{E}_2(r)\ L_1(r,r')\ \mathcal{E}_1(r') = \int_0^{r_o} dr\ r\ \mathcal{E}_1(r')L_1(r,r')\mathcal{E}_2(r) = $$
$$= \int_0^{r_o} dr'\ r'\mathcal{E}_1(r')\ L_4(r',r)\mathcal{E}_2(r) = \int_0^{r_o} dr'\ r'\ \mathcal{E}_2(r)L_4(r',r)\mathcal{E}_1(r') \qquad \text{(B-5)}$$

$$\int_0^{r_o} dr' r'\ \mathcal{E}_2(r')L_1(r',r)\mathcal{E}_1(r) = \int_0^{r_o} dr' r'\ \mathcal{E}_1(r)\ L_1(r',r)\mathcal{E}_2(r') = $$
$$= \int_0^{r_o} dr\ r\ \mathcal{E}_1(r)L_4(r,r')\mathcal{E}_2(r') = \int_0^{r_o} dr\ r\ \mathcal{E}_2(r')L_4(r,r')\mathcal{E}_1(r). \qquad \text{(B-6)}$$

Now, a first-order variation of Λ given by the expression (36) with respect to the simultaneous variation of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ gives the following result:

$$L_0^2\delta\Lambda + 2L_0\Lambda\left[\int_0^{r_o} dr\ J_1(\eta_1 r)\delta\mathcal{E}_1(r) + \int_0^{r_o} dr\ r\ \frac{d}{dr}J_1(\eta_1 r)\delta\mathcal{E}_2(r)\right] =$$
$$= \int_0^{r_o} dr\ r\ \delta\mathcal{E}_1(r)L_3(r,r')\mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_1(r)L_3(r,r')\delta\mathcal{E}_1(r') +$$
$$+ \int_0^{r_o} dr\ r\ \delta\mathcal{E}_2(r)L_2(r,r')\mathcal{E}_2(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_2(r)L_2(r,r')\delta\mathcal{E}_2(r') +$$
$$+ \int_0^{r_o} dr\ r\ \delta\mathcal{E}_2(r)L_1(r,r')\mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_2(r)L_1(r,r')\delta\mathcal{E}_1(r') +$$
$$+ \int_0^{r_o} dr\ r\ \delta\mathcal{E}_1(r)L_4(r,r')\mathcal{E}_2(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_1(r)L_4(r,r')\delta\mathcal{E}_2(r') +$$

or

$$L_0^2\delta\Lambda = \int_0^{r_o} dr\ r\ \delta\mathcal{E}_2(r)\left[-2L_0\Lambda\frac{d}{dr}J_1(\eta\ r) + L_2(r,r')\mathcal{E}_2(r') + \right.$$
$$\left. + L_1(r,r')\mathcal{E}_1(r')\right] + \int_0^{r_o} dr\ r\ \delta\mathcal{E}_1(r)\left[-2L_0\Lambda\frac{J_1(\eta_1 r)}{r} + \right.$$
$$\left. + L_3(r,r')\mathcal{E}_1(r') + L_4(r,r')\ \mathcal{E}_2(r')\right] +$$

$$+ \int_0^{r_o} dr\ r\ \mathcal{E}_1(r) L_3(r,r') \delta\mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_2(r) L_2(r,r') \delta\mathcal{E}_2(r')$$

$$+ \int_0^{r_o} dr\ r\ \mathcal{E}_2(r)\ L_1(r,r') \delta\mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_1(r) L_4(r,r') \delta\mathcal{E}_2(r') \qquad \text{(B-7)}$$

Now, by virtue of the properties (B-1), (B-2), (B-5) and (B-6), the following relations can be established;

$$\int_0^{r_o} dr\ r\ \mathcal{E}_1(r) L_3(r,r') \delta\mathcal{E}_1(r') = \int_0^{r_o} dr\ r\ \delta\mathcal{E}_1(r) L_3(r,r') \mathcal{E}_1(r'), \qquad \text{(B-8)}$$

$$\int_0^{r_o} dr\ r\ \mathcal{E}_2(r) L_2(r,r') \delta\mathcal{E}_2(r') = \int_0^{r_o} dr\ r\ \delta\mathcal{E}_2(r) L_2(r,r') \mathcal{E}_2(r'), \qquad \text{(B-9)}$$

$$\int_0^{r_o} dr\ r\ \mathcal{E}_2(r) L_1(r,r') \delta\mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \mathcal{E}_1(r) L_4(r,r') \delta\mathcal{E}_2(r') =$$

$$= \int_0^{r_o} dr\ r\ \delta\mathcal{E}_2(r) L_1(r,r') \mathcal{E}_1(r') + \int_0^{r_o} dr\ r\ \delta\mathcal{E}_1(r) L_4(r,r') \mathcal{E}_2(r'). \qquad \text{(B-10)}$$

In view of (B-8) to (B-10) together with relations (27) and (28), it follows readily that the right-hand side of (B-7) vanishes. Therefore, since $L_0 \neq 0$, we have $\delta\Lambda = 0$; i.e., Λ given by (36) is stationary with respect to the first-order simultaneous variation of $\mathcal{E}_1(r)$ and $\mathcal{E}_2(r)$ about their respective correct values.

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